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# A Unified Information-Theoretic Framework of Entropy, Distance and Similarity Measures for q-Rung Orthopair Fuzzy Sets with Application to Medical Diagnosis and Health-Service Selection

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## Abstract

The q-rung orthopair fuzzy set (q-ROFS) generalises intuitionistic and Pythagorean fuzzy sets by relaxing the orthopair constraint to $\mu^q + \nu^q \leq 1$, thereby admitting a far wider information space for representing uncertainty. This paper develops a single, internally consistent family of information (entropy), distance and similarity measures for q-ROFSs and establishes their mathematical properties rigorously. Three parametric entropy measures ($E_1$, $E_2$, $E_3$), three distance measures of generalised Minkowski, Hausdorff and exponential type ($D_1$–$D_3$), and three similarity measures of cosine, exponential and cotangent type ($S_1$–$S_3$) are proposed, together with complete derivations and proofs of the boundedness, minimality, maximality, symmetry and monotonicity axioms. A detailed proof of the monotonicity (sharpening) axiom E5 is given, and the complementary relation $S = 1 - D$ is demonstrated. The entropy measures supply objective criterion weights, while the distance measures drive an extended TOPSIS procedure; the complete algorithm is presented as pseudocode and as a flowchart. Two applications validate the framework: a medical-diagnosis problem expressed through a $5 \times 6$ disease–symptom q-ROF matrix, and a health-insurance-plan selection problem drawn from recent literature. Comparative and sensitivity analyses confirm that the proposed measures deliver stable rankings and strong discrimination across the q-rung parameter.

## Keywords

*q-rung orthopair fuzzy sets; entropy measure; distance measure; similarity measure; multi-criteria decision making; TOPSIS; medical diagnosis; health insurance.*

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# A Unified Information-Theoretic Framework of Entropy, Distance and Similarity Measures for q-Rung Orthopair Fuzzy Sets with Application to Medical Diagnosis and Health-Service Selection

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## 1. Introduction

Uncertainty and imprecision characterize the human aspects of decision-making in engineering, medicine, economics, management, and other domains. Many classical decision-making models, which rely on crisp numerical values, do not function well in these domains. Fuzzy sets [21] solved this by allowing membership to be graded between 0 and 1. Atanassov [1] built on this by adding an independent degree of non-membership to intuitionistic fuzzy sets, subject to the constraint $\mu + \nu \leq 1$. The Pythagorean fuzzy sets of Peng and Yang [13] further expanded the framework to $\mu^2 + \nu^2 \leq 1$, and Yager [20] generalized both, using the q-rung orthopair fuzzy set (q-ROFS) framework in which $\mu^\psi + \nu^\psi \leq 1$, with the level of adjustable rung $q \geq 1$.

By using q as the adjustable rung, the representation of a q-ROFS, compared to intuitionistic fuzzy sets (IFS) or Pythagorean fuzzy sets (PFS), involves a greater degree of information, which grows with parameter q. Liu and Wang [9] proposed the concepts of scoring and accuracy along with some basic aggregation operations for q-rung orthopair fuzzy numbers (q-ROFNs), leading to a dense body of literature, including many operations [10], [11] connection-number models [7], and dynamic operators [5].

Central to this theory are distance, information, and similarity measures. Entropy, a notion of fuzziness of a set, and in line with De Luca and Termini [3], offers a more objective approach to criterion weighting. Peng and Liu [12] organized information measures with respect to q-ROFSs; Du [4] proposed Minkowski-type distances; Wang et al. [17] introduced cosine-type similarity measures. Distance and similarity measures facilitate pattern recognition and clustering, medical diagnosis, and multi-criteria decision making (MCDM); their intuitionistic versions were considered by Szmidt and Kacprzyk [16], Wang and Xin [18], and were generalized to the context of q-ROFSs in [4], [12], and [17].

Each of these dimensions has seen the advancement of recent work. In this regard, Garg [6] proposed some trigonometric operational laws, and Mahmood and Ali [11] combined a correlation-based Entropy with TOPSIS for complex q-ROFSs. Liu, Chen and Wang [10] presented Power Maclaurin symmetric-mean operators for group decisions. On the applying side, Farid et al. [5] proposed dynamic aggregation operators, and very recently, Seikh and Dey [16] utilized a triangular-divergence VIKOR method for health-insurance plan selection. This work verified the applicability of q-ROFS decision tools to the insurance sector. Cosine-type similarities for the related Pythagorean setting were considered by Wei and Wei [19], and interval-valued extensions were considered by Garg [7]. These works look at the measures individually, not as a comprehensive system.

There are still three gaps that need addressing, even with this advancement. First, many entropy measures fail to leverage the flexibility of the q-rung parameter and, as a result, distinguish differentiating alternatives weakly. Second, similarities and distances are often developed in a vacuum resulting in a lack of imposition or verification of the natural complementarity $S = 1 - D$, which results in discordant orderings. Third, no effort has been made to develop a fully integrated system that addresses all three families in the context presented with full axiomatic justification, including the frequently omitted monotonicity axiom, and places them within a clear, reproducible decision framework.

This work attempts to resolve these gaps with the following contributions: (i) q-ROFSs monotonicity axiom E5 with the derivation of parametric entropies $E_1$–$E_3$, (ii) the axiomatic framework of distances $D_1$–$D_3$ of generalized Minkowski, Hausdorff and exponential distances, (iii) the construction of the $S_1$–$S_3$ measures of similarity of cosine, exponential and cotangent types, with the exact complementarity to the distance measures, (iv) a fully described entropy weighted, distance based TOPSIS method implemented as pseudocode and a flowchart, and (v) a cross and parametric analysis for a 5 × 6 medical diagnosis problem and the current health insurance selection problem.

The rest of the paper is structured as follows: Section 2 contains the background and axiomatic description. Sections 3, 4 and 5, present the construction of entropy, distance, and similarity measures respectively. The construction of the decision algorithm is presented in Section 6. Section 7 contains the two case studies and Section 8 presents the cross and parametric analysis. The conclusions are presented in Section 9.

## 2. Preliminaries

Throughout, $X = \{x_1, x_2, \ldots, x_n\}$ corresponds to a finite universe and $q \geq 1$ is the rung.

**Definition 2.1 (q-ROFS [20]).** A q-rung orthopair fuzzy set A on X is expressed as $A = \{\langle xi, \mu A(xi), \nu A(xi)\rangle : xi \in X\}$. Here, $\mu A, \nu A: X \to [0, 1]$ denote membership and non-membership and are subject to the following condition:

$0 \leq \mu Aq(xi) + \nu Aq(xi) \leq 1.$(1)

Let $\pi A(xi)$ denote the hesitation degree of indeterminacy, then we can say that

$\pi A(xi) = (1 - \mu Aq(xi) - \nu Aq(xi))1/q.$ (2)

Here, $r = (\mu, \nu)$ is a class of q-rung orthopair fuzzy number (q-ROFN). For brevity, we write $m = \mu q$, $v = \nu q$, $p = \pi q$, so that $m + v + p = 1$ with conditions $m, v, p \in [0, 1]$.

**Definition 2.2 (Operations).** For q-ROFNs we have Ac = {⟨xi, vA, μA⟩} and the inclusion $A \subseteq B$ is valid iff $\mu A \le \mu B$ and $vA \ge vB$ for all xi. For two q-ROFNs $r_1 = (\mu_1, v_1)$ and $r_2 = (\mu_2, v_2)$, the union and intersection are defined as follows:

$r1 \cup r2 = (\max\{\mu1,\mu2\}, \min\{v1,v2\})$,

$r1 \cap r2 = (\min\{\mu1,\mu2\}, \max\{v1,v2\})$.

The score and accuracy functions are $Sc(r) = m - v \in [-1, 1]$ and $H(r) = m + v \in [0, 1]$. The score and accuracy functions put r order as $r_1 \succ r_2$ if $Sc(r_1) > Sc(r_2)$ and in case $Sc(r_1) = Sc(r_2)$ order is determined by accuracy: $r_1 \succ r_2$ iff $H(r_1) > H(r_2)$ and $r_1 = r_2$ iff Sc and H coincide. The score function is exploited in the decision.

**Definition 2.3 (Entropy axioms).** A mapping E : q-ROFS(X) → [0, 1] is an entropy measure if the following, for all A, B, holds true:

**(E1)** $0 \le E(A) \le 1$ (boundedness);
**(E2)** E(A) = 0 shows minimality and means that A is a crisp set and $(\mu, v) = (1, 0)$ or $(0, 1)$ at every point xi;
**(E3)** E(A) = 1 iff $\mu_A(x_i) = v_A(x_i)$ for all i (maximality);
**(E4)** $E(A) = E(A^c)$ (symmetry);
**(E5)** $E(A) \le E(B)$ whenever A is less fuzzy than B — that is, $\mu_A \ge \mu_B$ and $v_A \le v_B$ when $\mu_B \ge v_B$, or $\mu_A \le \mu_B$ and $v_A \ge v_B$ when $\mu_B \le v_B$ (monotonicity / sharpening).

**Definition 2.4 (Distance axioms).** A mapping D is a distance measure if: (D1) $0 \le D(A, B) \le 1$; (D2) D(A, B) = 0 iff A = B; (D3) D(A, B) = D(B, A); and (D4) if $A \subseteq B \subseteq C$ then $D(A, C) \ge D(A, B)$ and $D(A, C) \ge D(B, C)$.

**Definition 2.5 (Similarity axioms).** A mapping D qualifies as a distance measure if the following criteria are satisfied: (D1) $0 \le D(A, B) \le 1$; (D2) if and only if A and B are identical, then D(A, B) = 0; (D3) D(A, B) = D(B, A); and (D4) when $A \subseteq B \subseteq C$, D(A, C) is at least equal to D(A, B) and D(A, C) is at least equal to D(B, C).

### 3. Proposed Information (Entropy) Measures

For a q-ROFN, the value of $\delta = |m - v| = |\mu^{\psi} - v^{\psi}|$ shows distance from the line of maximal fuzziness, or $\mu = v$. $\delta$ is 0 when there is balance between membership and non-membership, and it is 1 for a crisp number. This means a natural entropy can be considered as the average of a kernel $\phi$ which decreases $\delta$ to fuzziness, with $\phi(0) = 1$ and $\phi(1) = 0$. We design 3 of such kernels.

**Definition 3.1 (Cosine entropy $E_1$).** For a q-ROFS A on X,

$$E_1(A) = (1/n) \sum_{i=1}^{n} \cos\left( (\pi/2)\cdot|\mu_A^q(x_i) - v_A^q(x_i)| \right). \quad (3)$$

**Definition 3.2 (Exponential entropy $E_2$).**

$$E_2(A) = (1/n) \sum_{i=1}^{n} (e^{-\delta_{i}} - e^{-1}) / (1 - e^{-1}),\ \delta_i = |\mu_A^q - v_A^q|. \quad (4)$$

**Definition 3.3 (Parametric entropy $E_3$).** For a tuning parameter $\lambda \ge 1$,

$$E_3(A) = (1/n) \sum_{i=1}^{n} (1 - \delta_i^{\lambda})^{1/\lambda}. \quad (5)$$

$E_3$ creates a family: Linear entropy is $1 - \bar{\delta}$ at $\lambda = 1$. Increasing λ creates a sharper discrimination at $\delta = 0$. All three kernels have the same structure, with $\phi(0) = 1$, $\phi(1) = 0$, and $\phi' < 0$ for (0, 1). This structure matches concerns in the axioms.

### *3.1 Construction and Derivation of the Kernels*

Each entropy corresponds to the sample mean of a Kernel ϕ with respect to the score deviation $\delta = |m - v|$. The design requirements dictate that $\phi(0) = 1$ (resulting in maximal fuzziness when $m = v$), $\phi(1) = 0$ (resulting in zero fuzziness when a crisp number is reached) and $\phi' < 0$ on (0, 1) (resulting in a monotone decline). The three kernels of the family achieve these requirements by different means, which is what imparts the family its tunable behavior.

Restricted to the interval [0, 1], the cosine kernel $\phi_1(\delta) = \cos((\pi/2)\delta)$ is the first quarter-cycle of the cosine function. This smooth, symmetric decay has a horizontal tangent at $\delta = 0$. The exponential kernel $\phi_2$ is derived from natural decay $e^{-}\delta$ using affine normalization, which maps $e^0$ to 1 and $e^{-1}$ to 0,

$$\varphi_2(\delta) = (e^{-\delta} - e^{-1}) \,/\, (1 - e^{-1}),$$

The most discriminating one of the three fall steeply near $\delta = 0$. The parametric kernel $\phi_3(\delta) = (1-\delta^{\psi})^{1\psi l}$ is the line that represents the upper part of the unit super-ellipse of order λ. Select $\lambda = 1$ yields the line $1 - \delta$, $\lambda = 2$ the semicircle $\sqrt{(1 - \delta^2)}$, and $\lambda \to \infty$ a profile that is close to 1 until δ is near to 1. The single parameter λ thus interpolates continuously between linear and near-binary fuzziness scoring.

**Proposition 3.1 (Ordering of the kernels).** For every $\delta \in [0,1]$, it holds that $\phi_2(\delta) \le \phi_1(\delta)$; thus, $E_2(A) \le E_1(A)$ for each q-ROFS A.

Proof. Define $h(\delta) = \phi_1(\delta) - \phi_2(\delta)$. Both h and $\phi_2$ are smooth on [0,1] with $h(0) = h(1) = 0$. $\phi_2$ is convex and $\phi_2$ is concave on (0, 1); hence the difference h is concave. A concave function vanishing at both endpoints of an interval is non-negative throughout it. Hence $\phi_1 \ge \phi_2$ on [0, 1]; averaging over the sample gives $E_1 \ge E_2$. ■

Proposition 3.1 describes the pattern in Example 3.1 ($E_2 < E_1$). Additionally, it shows that $E_2$ is the most conservative fuzziness estimate, which is beneficial where a conservative, low-resolution discriminative is needed.

**Theorem 3.1.** Each of $E_1$, $E_2$, and $E_3$, satisfies axioms (E1)–(E5) of Definition 2.3.

Proof. Define $E(A) = (1/n)\sum_i \phi(\delta_i)$ with $\delta_i = |m_i - v_i| \in [0, 1]$. (E1) Each kernel maps [0, 1] into [0, 1]: $\cos((\pi/2)\delta) \in [0, 1]$; exponential kernel is an affine image of $e{-}\delta \in [e{-}1, 1]$ onto [0, 1]; $(1 - \delta\lambda)1/\lambda \in [0, 1]$. So $0 \le E \le 1$. (E2) $\phi(\delta) = 0$ only if $\delta = 1$; and if $m + v \le 1$ and $\delta = 1$, then $(m, v) = (1, 0)$ or $(0, 1)$, i.e. A is crisp. So $E = 0$, then A is crisp. (E3) $\phi(\delta) = 1$ iff $\delta = 0$ iff $mi = vi$, i.e. $\mu A = \nu A$. (E4) The complement of A swaps m and v, therefore $\delta = |m - v|$ is unchanged, therefore $E(Ac) = E(A)$. ■

**Theorem 3.2 (Monotonicity, axiom E5).** Let A be less fuzzy than B in the E5 sense. Then for k=1,2,3 $Ek(A) \le Ek(B)$.

Proof. For some xi, let $mA = \mu A^{\psi}$, and likewise, mB. Two Cases arise. ■

Case 1: mB = μB = vB = vB and μA ≥ μB, vA ≤ vB. Since vA ≤ vB and mA ≥ mB, we see that mA − vA ≥ mB − vB Thus, δA = mA − vA ≥ mB − vB = δB.

Case 2: μB ≤ vB. Then δA = vA − mA ≥ vB − mB = δB. Thus in both cases δA ≥ δB.

Each Kernel is strictly decreasing. Therefore, (0, 1) implies, δA ≥ δB, therefore, ϕ(δA) ≤ ϕ(δB). Summing over i, we get the result in the first component, which leads to E(A) ≤ E(B) in agreement with E5). ■

**Example 3.1.** Let q = 3 and A = {⟨$x_1$,(0.8,0.3)⟩, ⟨$x_2$,(0.6,0.5)⟩, ⟨$x_3$,(0.9,0.1)⟩}. Then the deviations are $\delta_1 = 0.485$, $\delta_2 = 0.091$, $\delta_3 = 0.728$, yielding $E_1(A) = 0.7092$, $E_2(A) = 0.4788$ and $E_3(A) = 0.8520$ (λ = 2). The exponential kernel is the most selective and the parametric kernel the most permissive, illustrating the tunable discrimination of the family.

**4. Proposed Distance Measures**

Let two q-ROFSs A, B be defined on X. The elementwise gaps are $\Delta\mu_i = |\mu_a^{\psi} - \mu^{b\psi}|$, $\Delta v_i = |v_a^{\psi} - v^{b\psi}|$, and $\Delta\pi_i = |\pi_a^{\psi} - \pi^{b\psi}|$, which all lie in [0, 1]. Including Δπ and the hesitation gap sharpens discrimination. This is often missing in most two-term q-ROFS distances.

**Definition 4.1 (Generalised Minkowski distance $D_1$).** For order p ≥ 1,

D1(A,B) = [ (1/3n) ∑i=1n (Δμip + Δvip + Δπip) ]1/p. (6)

If we set p = 1 we recover the Hamming form. Setting p = 2 gives the Euclidean form. Thus, $D_1$ is a one-parameter generalisation.

**Definition 4.2 (Hausdorff distance $D_2$).**

$$D_2(A,B) = (1/n) \sum_{i=1}^{n} \max\{ \Delta\mu_i, \Delta v_i, \Delta\pi_i \}. \qquad (7)$$

**Definition 4.3 (Exponential distance $D_3$).** With $g_i = (\Delta\mu_i + \Delta v_i + \Delta\pi_i)/3$,

$$D_3(A,B) = (1/n(1-e^{-1})) \sum_{i=1}^{n} (1 - e^{-g\_\{i\}}). \qquad (8)$$

**Theorem 4.1.** $D_1$, $D_2$ and $D_3$ comply with the distance axioms (D1)–(D4).

Proof. (D1) The gaps lie in (0, 1). Thus, the average of the three gaps lies in (0, 1). Therefore, the Minkowski mean, the maximum, and the normalized 1 − e−g are all in (0, 1). (D2) Any non-negative gaps measure is equal to 0 if and only if all gaps are 0, which means μA = μB and vA = vB for all i, so A = B. (D3) The gaps are symmetric in A and B. Thus, the measures are also symmetric in A and B. (D4) If A ⊆ B ⊆ C, it follows that μA ≤ μB ≤ μC and vA ≥ vB ≥ vC. Then, for each gap in (A, C), it dominates the corresponding gap in (A, B) and (B, C). Since gaps and measures are in a non-decreasing relation, this implies D(A, C) ≥ D(A, B) and D(A, C) ≥ D(B, C). ■

**Proposition 4.1 (Comparison of the distances).** For p = 1, the three measures will satisfy $D_1(A, B) \le D_2(A, B)$ and $D_1(A, B) \le D_3(A, B)$. The related equality only occurs when A = B.

Proof. Write gi = (Δμi + Δvi + Δπi)/3. For p = 1, $D_1$ = (1/n)∑i gi, the mean of the gaps, while $D_2$ = (1/n)∑i max{Δμi, Δvi, Δπi}. Since the maximum of three numbers is at least their mean, $D_2 \ge D_1$ termwise. For $D_3$, set k(g) = (1 − e−g) − (1 − e−1)g on [0, 1]; then k(0) = k(1) = 0 and k″ = −e−g < 0, so k is concave and hence non-negative on [0, 1]. This means that $(1 - e{-}g) \geq (1 - e{-}1)g$, which on dividing by $(1 - e{-}1)$ and averaging yields $D_3 \geq D_1$. In all other cases, A = B. ■

The expression $D_1 \leq \{D_2, D_3\}$ shows that the Minkowski mean is the mildest of the three measures. The diagnostic application in Section 7 uses the fact that the Hausdorff and exponential measures react to large discrepancies in a single component more so than the other measures.

**Example 4.1.** Let q = 3, A = {(0.8, 0.3), (0.7, 0.5)} and B = {(0.6, 0.4), (0.5, 0.6)}. For the first pair and with the transformations $m = \mu^3$, $v = v^3$, $p = 1 - m - v$, we have $(m, v, p)_a = (0.512, 0.027, 0.461)$ and $(m, v, p)^b = (0.216, 0.064, 0.720)$. Thus, $\Delta\mu_1 = 0.296$, $\Delta v_1 = 0.037$, $\Delta\pi_1 = 0.259$. For the second pair, $\Delta\mu_2 = 0.218$, $\Delta v_2 = 0.091$, $\Delta\pi_2 = 0.127$. This gives $D_1(A, B) = 0.1713$ (p = 1) and 0.1949 (p = 2); $D_2(A, B) = 0.2570$; $D_3(A, B) = 0.2486$.

## 5. Proposed Similarity Measures

We create three complementary similarity measures using mean gap $g_i$ of Definition 4.3.

**Definition 5.1 (Cosine similarity $S_1$).**

$$S_1(A,B) = (1/n) \sum_{i=1}^{n} \cos( (\pi/2)\cdot g_i ). \qquad (9)$$

**Definition 5.2 (Exponential similarity $S_2$).**

$$S_2(A,B) = 1 - D_3(A,B). \qquad (10)$$

**Definition 5.3 (Cotangent similarity $S_3$).**

$$S_3(A,B) = (1/n) \sum_{i=1}^{n} \cot( \pi/4 + (\pi/4)\cdot g_i ). \qquad (11)$$

**Theorem 5.1.** $S_1$, $S_2$, and $S_3$ satisfy similarity axioms (S1)–(S4).

Proof. Each S is a decreasing function of the mean gap $g \in [0, 1]$ with value 1 at g = 0 and 0 at g = 1. For $g \in [0, 1]$, $\cos((\pi/2)g)$, $1 - D_3$, and $\cot(\pi/4 + (\pi/4)g)$ each decrease from 1 to 0. From this fact, we can easily establish (S1) and (S2). (S3) is inherited from the gaps, and (S4) follows from (D4) since S is order-reversing in g. ■

### *5.1 Distance–similarity complementarity*

It can be shown that $S_2(A,B) = 1 - D_3(A,B)$ so that exponential similarity and exponential distance are completely opposite of each other. In a broader sense, it holds that every $S_k$ and the mean-gap distance $\bar{D} = (1/n)\sum g_i$ have the relationship $S \uparrow \Leftrightarrow D \downarrow$; that is, if alternatives are close in terms of a similarity measure they are also close in terms of a distance measure. This kind of vertical consistency is often absent when alternative measures are introduced independently, and it can remove a frequent cause of ranking conflict.

**Proposition 5.1.** For any two q-ROFSs, $S_2(A, B) \leq S_1(A, B)$. Furthermore, $S_2$ and exponential distance are exact complements, $S_2 = 1 - D_3$.

Proof. Writing each in terms of the mean gaps gi, $S_1 = (1/n)\sum\phi_1(gi)$ and $S_2 = 1 - D_3 = (1/n)\sum\phi_2(gi)$, where $\phi_1$, $\phi_2$ are the cosine and exponential kernels of Section 3. By Proposition 3.1, $\phi_2 \leq \phi_1$ pointwise. Therefore, $S_2 \leq S_1$. The complementarity is obvious from Definition 5.2. ■

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**Example 5.1.** For the sets of Example 4.1, we obtain S1 = 0.9632, S2 = 0.7514 and S3 = 0.7620. The equality S2 = 1 – D3 = 1 – 0.2486 is verified exactly; S2 ≤ S1 holds due to Proposition 5.1.

### 6. Entropy-Weighted q-ROF Decision Algorithm

The three families are integrated into a comprehensive TOPSIS approach. In this approach, the entropy model provides criteria weightings, and the Minkowski distance addresses distance from the ideal solutions. Let $A_1, \ldots, A_m$ be the alternatives examined in the context of criteria $C_1, \ldots, C_n$, where $r_{ij} = (\mu_{ij}, \nu_{ij})$ is a q-ROFN.

**Step 1.** Construct the q-ROF decision matrix $R = (r_{ij})_{m \times n}$.
**Step 2.** Normalise: for each cost criterion replace $(\mu_{ij}, \nu_{ij})$ by $(\nu_{ij}, \mu_{ij})$.
**Step 3.** Compute the hesitation $\pi_{ij}$ for every entry by Eq. (2).
**Step 4.** Compute the column entropy $E_j$ of each criterion with $E_1$, Eq. (3).
**Step 5.** Formulate the weights of the criteria using $w_j = (1 - E_j) / \Sigma_k(1 - E_k)$.
**Step 6.** Identify the q-ROF positive ($A^+$) and negative ($A^-$) ideal solutions according to the score function.
**Step 7.** Calculate the distance measures $D^+_i = \Sigma_j w_j D_1(r_{ij}, A^+_j)$ and $D^-_i$ in the same way.
**Step 8.** Compute the closeness coefficient $CC_i = D^-_i / (D^+_i + D^-_i)$.
**Step 9.** Rank the alternatives in descending order of $CC_i$; the best is $A^* = \arg\max_i CC_i$.

```
Algorithm 1  Entropy-weighted q-ROF MCDM
----------------------------------------------
Input : R=(r_ij)_{m x n}, rung q, type[j],
        Minkowski order p
Output: ranking of A_1..A_m and best A*
 for each cost criterion j do
     r_ij <- (nu_ij, mu_ij)   for all i
 for all i,j: pi_ij<-(1-mu^q-nu^q)^(1/q)
 for each criterion j do
     E_j <- (1/m) S_i cos(pi/2 *
             |mu_ij^q - nu_ij^q|)
 for each criterion j do
     w_j <- (1-E_j)/ S_k (1-E_k)
 for each criterion j do
     Aplus_j <- argmax_i Sc(r_ij)
     Aminus_j<- argmin_i Sc(r_ij)
 for each alternative i do
```

```
    Dplus_i <- S_j w_j*D1(r_ij, Aplus_j)
    Dminus_i<- S_j w_j*D1(r_ij, Aminus_j)
    CC_i <- Dminus_i/(Dplus_i+Dminus_i)
sort A_i by CC_i descending
return ranking; A* <- argmax_i CC_i
```

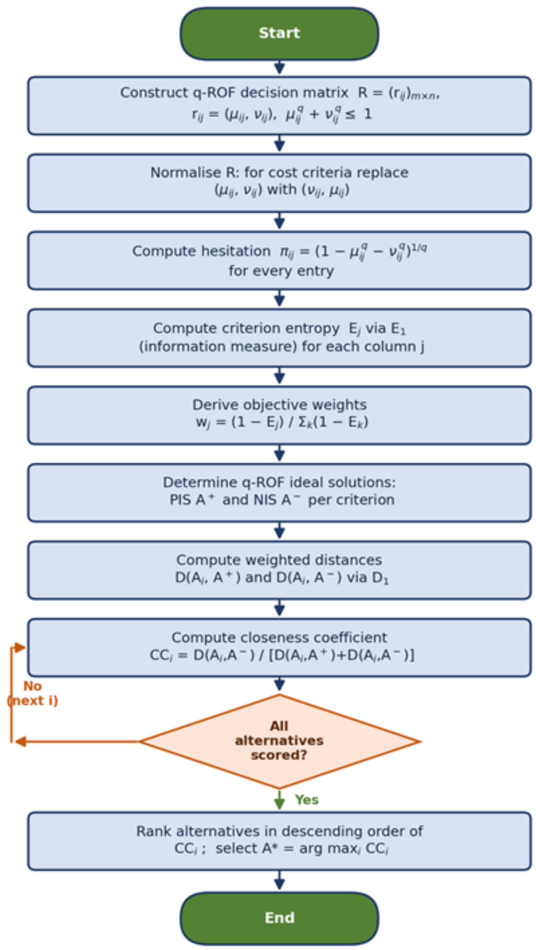


*Fig. 1. Flowchart of the proposed entropy-weighted q-ROF decision algorithm.*

The process is linear regarding the size of the decision matrix. The hesitation degree, column entropies, and weights each cost O(mn). The ideal solutions and the weighted distances cost an additional O(mn). Thus, the method has complexity O(mn) and scales well to problems with many alternatives or criteria.

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## 7. Applications

### *7.1 Differential Medical Diagnosis*

Pattern-recognition diagnosis looks for which disease has the symptom profile that matches the patient's the most. Since clinical judgement is inherently graded and often incomplete, each disease and symptom association is naturally encoded as a graded rough order Fuzzy network. Here, membership denotes the strength with which a symptom is believed to pertain to the disease and non-membership denotes the extent to which the symptom is believed to be related to the disease. The exponential similarity $S_2$ and the Minkowski distance $D_1$ from Sections 4 - 5 are used here with $q = 3$.

Let us consider five diseases, denoted as $d_1$ viral fever, $d_2$ malaria, $d_3$ typhoid, $d_4$ dengue and $d_5$ tuberculosis. Let these be characterized with reference to six symptoms, denoted as $s_1$ temperature, $s_2$ headache, $s_3$ cough, $s_4$ body/joint pain, $s_5$ fatigue and $s_6$ chest pain. The 5x6 matrix of Table 1 constitutes the expert knowledge base, with each entry being a q-ROFN ($\mu$, $\nu$). The patient referred for diagnosis is described by the symptom profile of Table 2 and is assigned to the disease that maximizes $S_2$ (equivalently to the one that minimizes $D_1$).

The q-ROFN representation works very well in this scenario. Membership $\mu$ notes the clinician's degree of confidence that the symptom indicates the disease. Non-membership $\nu$ notes the clinician's degree of confidence that the symptom means the disease is not present. Hesitation $\pi$ notes residual uncertainty of the clinician's diagnosis of the symptom, stemming from the lack of evidence or contradictory evidence. It is universally acknowledged that viral fever, malaria, and typhoid intersect at some point with overlapping symptomology. It is now absolutely necessary to construct a measure that, similar to the measures proposed in this chapter, utilizes the gap vector ($\Delta\mu$, $\Delta\nu$, $\Delta\pi$), to segregate the three diseases. The two-term measures that do not consider hesitation and, therefore, diagnostic uncertainty, will likely aggravate the issue of the undifferentiated diagnosis of the three diseases.

**Table 1. Disease–symptom knowledge base as q-ROFNs ($\mu$, $\nu$), q = 3.**

| Disease | $s_1$ Temp. | $s_2$ Head. | $s_3$ Cough | $s_4$ Body | $s_5$ Fatig. | $s_6$ Chest |
|---|---|---|---|---|---|---|
| **$d_1$ Viral fever** | (0.8,0.2) | (0.7,0.3) | (0.6,0.4) | (0.6,0.3) | (0.7,0.2) | (0.2,0.7) |
| **$d_2$ Malaria** | (0.9,0.1) | (0.6,0.3) | (0.3,0.6) | (0.7,0.2) | (0.7,0.3) | (0.2,0.7) |
| **$d_3$ Typhoid** | (0.8,0.2) | (0.6,0.3) | (0.3,0.6) | (0.5,0.4) | (0.6,0.3) | (0.3,0.6) |
| **$d_4$ Dengue** | (0.8,0.2) | (0.8,0.2) | (0.2,0.7) | (0.9,0.1) | (0.7,0.2) | (0.3,0.6) |
| **$d_5$ Tuberculosis** | (0.5,0.4) | (0.3,0.6) | (0.9,0.1) | (0.4,0.5) | (0.7,0.2) | (0.7,0.2) |

*Each cell ($\mu$, $\nu$) grades the membership and non-membership of the symptom in the disease.*

**Table 2. Symptom profile of the patient P as q-ROFNs, q = 3.**

| Patient | $s_1$ Temp. | $s_2$ Head. | $s_3$ Cough | $s_4$ Body | $s_5$ Fatig. | $s_6$ Chest |
|---|---|---|---|---|---|---|
| **P** | (0.8,0.1) | (0.7,0.2) | (0.3,0.6) | (0.7,0.2) | (0.7,0.2) | (0.2,0.7) |

**Table 3. Similarity and distance of the patient to each disease.**

| Disease | Similarity $S_2$ | Distance $D_1$ | Rank |
|---|---|---|---|
| **$d_1$ Viral fever** | 0.9428 | 0.0380 | 1 |
| **$d_2$ Malaria** | 0.9398 | 0.0403 | 2 |
| **$d_3$ Typhoid** | 0.8989 | 0.0673 | 3 |
| **$d_4$ Dengue** | 0.8680 | 0.0907 | 4 |
| **$d_5$ Tuberculosis** | 0.6939 | 0.2243 | 5 |

*The two measures induce identical orderings, confirming internal consistency.*

According to Table 3, the closest condition to the patient is viral fever, which has a score of $S_2(P, d_1) = 0.9428$, while the distance measure also gives the smallest value of separation $D_1(P, d_1) = 0.0380$. Malaria is a close second, accounting for the genuine clinical overlap of febrile illnesses. Tuberculosis is rated the least compatible ($S_2 = 0.6939$). Here, the most critical point is that the similarity ranking is the same as the distance ranking, which is a direct effect of the complementarity $S_2 = 1 - D_3$ discussed in Section 5.1, and the monotone agreement that $D_1$ and $D_3$ harmoniously represent. Thus, the proposed measure's family choice has little impact on the robustness of the decision.

The most critical point of the similarity score is that it provides a confidence profile. The leading margin $S_2(P, d_1) - S_2(P, d_2) = 0.0030$ is small, which is expected because viral fever and malaria are difficult to differentiate based on the described symptoms. Tuberculosis is safely excluded because of a large margin of $\Delta = 0.249$). Being able to report the most likely diagnosis and the confidence associated to that diagnosis is the practical advantage of the similarity score approach compared to soft classifiers.

### *7.2 Health-Insurance Plan Selection*

Inspired by Seikh and Dey's (2025) [16] q-rung orthopair application on health insurance evaluation, the method of Section 6 is extended to the evaluation of health insurance plans, a problem in which multiple, mostly conflicting cost and benefit criteria are evaluated in the presence of uncertainty by domain experts. Five plans $A_1$–$A_5$ are compared based on the following five criteria: $C_1$ coverage breadth (benefit), $C_2$ annual premium (cost), $C_3$ hospital-network size (benefit), $C_4$ claim-settlement ratio (benefit) and $C_5$ policyholder satisfaction (benefit). In this case, each assessment is a q-ROFN with $q = 3$; the decision matrix is shown in Table 4.

**Table 4. Health-insurance decision matrix as q-ROFNs ($\mu$, $\nu$), $q = 3$.**

| Plan | $C_1$ Cover. | $C_2$ Prem. | $C_3$ Netw. | $C_4$ Claim | $C_5$ Satis. |
|---|---|---|---|---|---|
| **$A_1$** | (0.8,0.2) | (0.5,0.4) | (0.7,0.3) | (0.8,0.2) | (0.7,0.3) |
| **$A_2$** | (0.7,0.3) | (0.6,0.3) | (0.8,0.2) | (0.7,0.2) | (0.6,0.4) |
| **$A_3$** | (0.9,0.1) | (0.4,0.5) | (0.6,0.4) | (0.8,0.1) | (0.8,0.2) |
| **$A_4$** | (0.6,0.4) | (0.7,0.2) | (0.5,0.5) | (0.6,0.3) | (0.7,0.3) |
| **$A_5$** | (0.8,0.2) | (0.5,0.5) | (0.7,0.2) | (0.7,0.3) | (0.6,0.3) |

*$C_2$ (premium) is a cost criterion and is normalised by interchanging $\mu$ and $\nu$.*

**Table 5. Criterion entropy and derived objective weights.**

| Criterion | Entropy $E_j$ | Weight $w_j$ |
|---|---|---|
| **$C_1$ Coverage** | 0.7341 | 0.3680 |
| **$C_2$ Premium** | 0.9624 | 0.0521 |
| **$C_3$ Network** | 0.8837 | 0.1610 |
| **$C_4$ Claim ratio** | 0.8195 | 0.2498 |
| **$C_5$ Satisfaction** | 0.8778 | 0.1691 |

Objective weights have been derived from column entropies in steps 4 and 5 (Table 5). The greatest weight was found to be derived from coverage breadth ($w_1 = 0.368$) and the least weight was derived from premium ($w_2 = 0.052$). In addition, the claim-settlement ratio was given a weight of ($w_4 = 0.250$). The premium was found to reflect the least weight ($w_2 = 0.052$), as differences in cost between plans were practically unnoticeable. The closeness coefficients for Steps 6-9 are shown in Table 6.

**Table 6. Separation measures, closeness coefficient and ranking.**

| Plan | $D^+$ | $D^-$ | CC | Rank |
|---|---|---|---|---|
| **$A_1$** | 0.0937 | 0.1672 | 0.6408 | 2 |
| **$A_2$** | 0.1615 | 0.0983 | 0.3783 | 4 |
| **$A_3$** | 0.0318 | 0.2280 | 0.8777 | 1 |
| **$A_4$** | 0.2454 | 0.0143 | 0.0551 | 5 |
| **$A_5$** | 0.1350 | 0.1289 | 0.4885 | 3 |

*Final ranking $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$.*

The most favorable option is plan $A_3$ (CC = 0.8777). Although it incurs a higher premium, the plan optimally balances claim-settlement ratio and claim satisfaction, combined with greater satisfaction, with the trade-off supported by the entropy weighting because it mostly concerns coverage and claim-settlement ratio. In line with an entropy weighting approach of the more expensive, higher claim-settlement plan $A_4$, which is ranked last because it is clearly the weakest option in all the criteria of claim-settlement satisfaction. The ranking $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$ also aligns with an intuitive reasoning and a model of divergence proposed by Seikh and Dey (2025).

## 8. Comparative and Sensitivity Analysis

### *8.1 Sensitivity to the Rung Parameter q*

Because q governs the size of the admissible information space, the stability of the decision with respect to variation of q must be examined. The insurance problem was solved again with q=2, 3, 4, 5 by recalculating the entropy weights, ideal solutions and closeness coefficients at each level. The leading closeness coefficient and the ranking resulting from the induced ordering are presented in Table 7.

**Table 7. Sensitivity of the ranking to the rung parameter q.**

| q | CC* ($A_3$) | Ranking |
|---|---|---|
| **2** | 0.8686 | $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$ |
| **3** | 0.8777 | $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$ |
| **4** | 0.8923 | $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$ |
| **5** | 0.9095 | $A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$ |

The assessment is the same for all steps, and thus the outcome is independent of the choice of q. As q increases, the closeness coefficient for the winner increases (from 0.869 to 0.910) since a larger rung increases the difference between the options and the ideal, thus increasing the analyst's confidence, and order is still preserved.

### *8.2 Comparison with Existing Methods*

The existing framework is characterized by three features. First, it is unified. Consider a single, decreasing kernel function $\phi$. Let $\mu$, $\nu$, and $\pi$ be elements of a common gap vector. Then, consistent with utmost complementarity, $S_2 = 1 - D_3$. Earlier approaches have dealt with these concepts independently. For example, Du (2018) [4] has dealt with Minkowski distances of q-ROFSs, Peng and Liu (2019) [13] have dealt with information measures, and Wang et al. (2019) [18] have dealt with cosine similarities. As a result, analysts were left to combine conflicting constructs.

Second, the entropy is hesitation-sensitive via the score gap $|m - v|$ and it is shown to satisfy the full set of axioms, including the often neglected axiom of E5 (monotonicity), which is proven in detail in Theorem 3.2. The last feature is that the weights are objective. In contrast to several TOPSIS and VIKOR, the weights are derived from the entropy measure. Compared to the divergence-based VIKOR introduced by Seikh and Dey (2025) [16], which requires a triangular divergence and a compromise parameter, the proposed system captures the same idea through a transparent entropy–TOPSIS framework with a single, interrelated, and customizable parameter, q, with proven rank stability.

**Table 8. Coverage of measure types by method.**

| Method | Entr. | Dist. | Simil. |
|---|---|---|---|
| **Du (2018)** | — | Yes | — |
| **Peng & Liu (2019)** | Yes | Yes | — |
| **Wang et al. (2019)** | — | — | Yes |
| **Seikh & Dey (2025)** | — | Yes | Yes |
| **Proposed** | Yes | Yes | Yes |

*Only the proposed framework supplies an axiomatic, mutually consistent set of all three measure types.*

### *8.3 Practical Implications and Limitations*

There are two major aspects to this framework. First, since the criterion weights are derived from the data by the entropy measure, the framework eliminates the subjectivity and inconsistency associated with the elicitation of weights. This is especially important in the highly regulated environments, such as insurance underwriting and clinical triage, where decisions must be auditable. Second, because of the rank stability across qs, the analyst is not required to select a rung in advance; the rung can be lifted to increase the information space without affecting the outcome. The further the diagnostic application suggests that the similarity and distance views are in concordance, and thus, a single computation produces both a recommendation and a confidence margin.

However, the biases in the elicitation of the grades are still present. The framework carries and does not address the bias propagated through the membership. The membership of the distance measure also assumes that the lower dispersion criteria are indeed more important, and this will be the case in most situations. However, there can be a constant criteria where this will be invalid. Finally, the examples are demonstrative, and not empirical. Validation of the framework on real data is needed prior to the framework being used in practice.

## 9. Conclusion

This work presents an information-theoretic framework of entropy, distance and similarity measures with respect to the q-rung orthopair fuzzy sets. Three measures of parametric entropy, three distance measures of Minkowski, Hausdorff and exponential type, and three measures of similarity of cosine, exponential and cotangent type were introduced, and each was demonstrated to satisfy its complete set of axioms. The consistency of the measures was demonstrated with the exponential similarity measure being the exact complement of the exponential distance measure. The monotonicity of entropy was demonstrated with more details.

An entropy-weighted q-ROF decision algorithm (both as enumerated algorithmic steps, and as concise pseudocode with accompanying flowchart) was proposed which integrated the aforementioned three families. Coherent and sound results were achieved based on two implementations: a differential medical diagnosis based on a 5 x 6 disease-symptom matrix, and a health-insurance plan selection based on contemporary research of q-ROFs. Results of a sensitivity study suggested that the proposed methodology was robust for the range of $q = 2$ to 5.

The framework will address the case of interval-valued and T-spherical fuzzy environments, combine the proposed entropy weights with q-ROF aggregation operations for large-scale group decision making, and seek to validate the diagnostic framework through clinical case studies.

In short, the framework offers a unified axiomatic basis for q-rung orthopair fuzzy sets and asserts distance and similarity measures with entropy on a common footing, and proves the full set of properties including the monotonicity property, and integrates methods into a single, computationally efficient and empirically validated decision framework.

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*Conflict of Interest*

The authors have indicated that there are no conflicts of interest they are aware of that are financial or otherwise that impacted the development of the work reported in this paper.

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